THE STRAIGHT LINE CONCEPT.

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INTRODUCTION.

The foundation of a science is the system of assumptions which gives precision to the concepts with which the science deals. It is essential that the system of assumptions together with the results obtained by applying the processes of logic to the concepts shall be free from contradiction. This freedom from contradiction is generally established by showing that the system of assumptions gives precision to some complete number system of arithmetic.

It is an important problem in any science to reduce the system of assumptions to a minimum. This problem is solved by excluding all assumptions which are logical consequences of other assumptions. When the system of assumptions of a science is reduced to a minimum the omission of any one assumption or the change of any one assumption will either lead to a contradiction or change the concepts of the science.

In order that a science shall not become a mere exercise in mental gymnastics the results obtained by applying the processes of logic to the concepts of the science must agree with observed results in the processes of the physical world.

The assumptions of a science are also called the axioms of a science. The assumptions of geometry are called axioms by Hilbert in the Grundlagen der Geometrie.

THE STRAIGHT LINE.

The logical entities with which rational geometry deals are the concepts named the point, the straight line and the plane. Precision is given to these concepts by axioms which have been arranged by Hilbert in five groups, called axioms of relation, axioms of order, axioms of congruence, axioms of parallels and axioms of continuity.

The importance of the straight line, whether by straight line we understand the intuitive entity of experience or the logical entity of rational geometry, depends primarily on the fact that a straight line is determined by any two of its points and can be indefinitely extended between any two of its points. Right here arises a question that can not be answered by experience or experiment. If the straight line is indefinitely extended beyond any two of its points, will there be found on the straight line two points at infinity, one point at infinity, or no point at infinity? This question, of course, can not be answered until precision has been given to the term distance.

In the plane determined by a given point and a given straight line, draw a straight line through the given point intersecting the given straight line and revolve the straight line about the given point. When the point of intersection of the revolving line with the given line moves to an infinite distance from the foot of the perpendicular from the given point to the given line, the revolving line is said to become parallel to the given line. In how many positions does the revolving line become parallel to the given line? This, again, is a question that can not be answered by experience or experiment. It can be answered only by the axiom of parallels.

If the axiom of parallels is made to read: Through a given point without a given line one and only one parallel to the line can be drawn — we have a geometry in which the straight line has only one point at infinity. This is the geometry of Euclid. Since the parabola meets the straight line at infinity in only one point, this geometry is also called the parabolic geometry.

If the axiom of parallels is made to read: Through a given point without a given line two and only two parallels to the line can be drawn — we have a geometry in which the straight line has two and only two points at infinity. Since the hyperbola intersects the straight line at infinity in two points, this geometry is called the hyperbolic geometry. The hyperbolic geometry was developed by Euclidean methods by Lobachevski and Bolyai.

If the axiom of parallels is made to read: Through a given point without a given line no parallel to the line can be drawn — we have a geometry in which the straight line has no point at infinity. Since the ellipse does not intersect the straight line at infinity in a real point, this geometry is called the elliptic geometry. The elliptic geometry has been discussed by Riemann, Clifford and Newcomb.

THE EXPRESSION FOR DISTANCE.

Much of the apparent mystery of hyperbolic and elliptic geometry vanishes when precision is given to the term distance. Distance is the result of measurement, and the measurement of a straight line requires that any part of the straight line may be applied anywhere along the straight line. If A, B, C are any three points in a straight line, and B is between A and C, the expression for distance must satisfy the equation

$$\text{distance } AB + \text{distance } BC = \text{distance } AC.$$

In a system of measurement introduced by Cayley in the Sixth Memoir on Quantics and developed by Klein, the expression for the distance between two points on a straight line is a function of the cross-ratio of these two points and two fixed points on the straight line. Let the fixed points be X, Y and A, B, C any three points taken in order on the straight line. By definition the cross-ratio of the four points A, B, X, Y is

$$(AX) / (AY) \div (BX) / (BY).$$

It follows from this definition that:

$$\text{cross-ratio } ABXY \times \text{cross-ratio } BCXY = \text{cross-ratio } ACXY.$$

Applying logarithms to this equation

$$\log \text{cross-ratio } ABXY + \log \text{cross-ratio } BCXY$$
$$= \text{log-cross-ratio } ACYX.$$

The expression

$$k \log \text{cross-ratio } ABXY$$

where k denotes any constant may therefore be taken as the expression for the distance between the points A, B.[1]

The pair of fixed points X, Y is called the absolute of linear measurement. When one of the points A, B coincides with a point of the absolute the distance AB becomes infinite. Hence when the absolute consists of two distinct real points, the straight line has two points at infinity; when the absolute consists of two coincident points, the straight line has one point at infinity; when

[1] The constant k must be so determined that the expression for distance has a real value. Since the logarithm is a many-valued function for which the series of values differ by multiples of $2\pi\sqrt{-1}$, when k is imaginary the expression for distance is a many-valued function for which the series of values differs by multiples of some real constant.

the absolute consists of a pair of imaginary points, the straight line has no point at infinity.

In the geometry of two dimensions the absolute must be the locus of the point pairs which are the absolute of all the lines in the plane. It follows that the points on the absolute are the points at infinity in the plane. By substituting in the equation of the absolute $f(x, y) = 0$ for x and y respectively $(x_1 + \lambda x_2)/(1 + \lambda)$ and $(y_1 + \lambda y_2)/(1 + \lambda)$ there will be found two values of λ, say λ_1 and λ_2, to which correspond the points of intersection of the straight line through the points (x_1, y_1) and (x_2, y_2) with the absolute, and the cross-ratio of the points (x_1, y_1), (x_2, y_2) and the points of intersection with the absolute is λ_1/λ_2. This cross-ratio is therefore readily calculated whether the points of intersection are real or imaginary.

If the equation of the absolute in homogeneous coördinates is $x^2 + y^2 - 4a^2t^2 = 0$, in order that the distance between two points within the absolute shall be real the constant k must be real. Every straight line determined by two points within the absolute has two points at infinity and we have the hyperbolic geometry of two dimensions. Points without the absolute are non-existent in this geometry.

If the equation of the absolute in homogeneous coördinates is $x^2 + y^2 + 4a^2t^2 = 0$ the constant k must be assumed imaginary in order that the distance between two points of the plane shall be real. The straight line has no point at infinity and we have the elliptic geometry. In this geometry the straight line has a finite length and must return into itself. The distance between two points has a series of values differing by multiples of the length of the entire straight line.

The points whose homogeneous coördinates are $x = 1$, $y = \sqrt{-1}$, $t = 0$; $x = 1$, $y = -\sqrt{-1}$, $t = 0$ satisfy the equations of the absolute in both the hyberbolic and elliptic geometries. These two points, named the imaginary circular points at infinity, constitute the absolute of plane parabolic geometry. The parabolic geometry is therefore a common limiting case of the hyperbolic and elliptic geometries. By a suitable choice of the constant k the parabolic geometry becomes the geometry of Euclid.

In the geometry of three dimensions the absolute of hyperbolic geometry may be written $x^2 + y^2 + z^2 - 4a^2t^2 = 0$; the absolute of elliptic geometry $x^2 + y^2 + z^2 + 4a^2t^2 = 0$; the absolute of parabolic geometry, $x^2 + y^2 + z^2 = 0$, $t = 0$, again a common limiting case of the absolute of hyperbolic and elliptic geometry.

By taking a in the equation of the absolute sufficiently large the hyperbolic and elliptic geometries approach identity with the parabolic geometry in finite regions of space, so that experience or experiment can never determine that the space of experience is hyperbolic, elliptic or parabolic.

The expression for the distance between two points must satisfy the requirement that the distance between two points shall be the same for all positions of the straight line on which the two points are located. A collinear motion of space into itself is represented analytically by a linear transformation which transforms the absolute into itself. The cross-ratio is an invariant of linear transformations. Hence the definition of distance $k \times$ log cross-ratio satisfies also this requirement of the expression for distance.

By the calculus of variations it is proved that in the elliptic, hyperbolic and parabolic geometries the straight line is the shortest distance between two points. Hilbert, by taking for absolute a triangle, has proved that the sum of two sides of a triangle may be equal to or less than the third side.

ANGLE MEASUREMENT.

In Cayley's system of measurement the measure of an angle is defined as a constant times the logarithm of the cross-ratio of the pencil of four rays formed by the sides of the angle and the tangents to the absolute from the vertex of the angle. If the equation of the absolute in line coördinates is $f(u, v) = 0$, the measurement of angles about the point of intersection of the lines (u_1, v_1) and (u_2, v_2) is analytically identical with the measurement of distance on a line through two points.

It follows from the definition that a right angle is an angle whose sides are harmonic conjugates with respect to the tangents from the angle vertex to the absolute. In the hyperbolic geometry any line through the pole of a given line with respect to the absolute intersecting the given line is perpendicular to it; the angle between lines intersecting on the absolute is zero, hence the two lines drawn from a given point to the intersections of a given line with the absolute are parallel to the given line; the sum of the angles of a triangle is less than 180°.

The chief attraction of hyperbolic geometry lies in the fact that one has the power to see the whole of hyperbolic space and to direct geometric constructions from a vantage point outside of this space. For example, to draw a common perpendicular to two straight lines, not intersecting and not parallel, connect by a straight line the poles of the given straight lines with respect to the absolute. This problem has been solved by Hilbert by methods such as a being living in hyperbolic space would be obliged to use. It is a simple matter to determine directly from the expression for distance that the locus of points in the hyperbolic plane equidistant from a given straight line is an ellipse tangent to the absolute where the given line meets the absolute.

GEOMETRY ON SURFACES OF CONSTANT TOTAL CURVATURE.

If R_1 and R_2 are the maximum and minimum radii of curvature of the normal sections of a curved surface at any point of the surface, the reciprocal of the product of R_1 and R_2 is called the Gaussian or total curvature of the surface at this point. The geometry of geodesics on surfaces whose total curvature is constant has striking analogies to plane Euclidean geometry. Euclid's definition of a straight line as a line which lies in the same manner with respect to all the points in the line, and his definition of a plane as a surface which lies in the same manner with respect to all straight lines in the plane, when taken in connection with Euclid's "Common Notions" implies the congruent displacement of a straight line into itself, that is the displacement of the straight line into itself such that any two points of the line may be made to coincide with any other two points of the line provided the distance between the first pair of points equals the distance between the second pair; and the congruent displacement of the plane into itself, that is the displacement of the plane into itself such that any portion of the plane bounded by straight lines may be made to coincide with any other portion provided the two portions are bounded by straight lines of equal length and the corresponding angles are equal. Now surfaces whose total curvature is constant and geodesics on these surfaces also possess this property of congruent displacement, provided displacement is suitably defined.

If the constant total curvature of the curved surface is $-a^2$, the geometry of geodesics on the curved surface is identical with the geometry of the plane with hyperbolic measurement.[1] The type of surfaces with constant negative total curvature is the pseudosphere of revolution, generated by revolving the tractrix about its asymptote.

If the constant total curvature of the curved surface is $+ a^2$, the geometry of geodesics on the curved surface is identical with the geometry of the plane with elliptic measurement.[1] The type of curved surfaces with constant positive total curvature is the sphere. It is important to note that the entire elliptic plane is represented on the hemisphere.

These statements show the reasonableness of using as equivalent the terms elliptic space and space of positive curvature; hyperbolic space and space of negative curvature; parabolic space and space of zero curvature.

CONTINUITY OF THE STRAIGHT LINE.

It remains to examine the elemental structure of the straight line. Adopting as definition of continuity the totality of all real numbers, is the totality of distances from a fixed point of the line to all other points of the line continuous? This question must be answered by establishing a correspondence between sets of numbers and points and lines, that is by a system of analytic geometry.

Let any pair of numbers (x, y) correspond to a point, any pair of numbers (u, v) correspond to a straight line, and let the equation $ux + vy + 1 = 0$ denote that the point (x, y) is on the line (u, v). The straight line is now determined by any two points and two straight lines intersect in only one point, that is, the straight line is the straight line of Euclid. If x, y and u, v are any numbers of the totality of numbers obtained from unity by applying a finite number of times the operations addition, subtraction, multiplication, division and taking the positive square root of unity plus the square of any number previously determined, Hilbert has proved that all the constructions of Euclid are possible. The straight line, however, is clearly not continuous, for no transcendental numbers occur in the totality of numbers represented on the straight line.

The continuity of the straight line is not a necessity of Euclid's

[1] Except for certain self-evident limitations due to the peculiarities of the surface.

geometry, it is not an intuitive property of the straight line and it cannot be proved by experiment. The continuity of the straight line can be established only by means of axioms, whether these axioms take the form given by Dedekind in his Essays on Number or by Hilbert in his Foundations of Geometry. When the continuity of the straight line has been established the Cartesian geometry at once follows.

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